Models of Set Theory I – Summer 2017

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Problem 25 [3 points] Fix a countable ground model M and a partial order $\mathbb{P} \in M$ that contains two incompatible conditions. Provide a counterexample to the following.

$$[p \Vdash \varphi \vee \psi] \text{ if and only if } [p \Vdash \varphi \ \vee \ p \Vdash \psi].$$

Problem 26 [4 points] Fix a countable ground model M and a partial order $\mathbb{P} \in M$. Let F be a filter on $\mathbb{P}$ which is not $\mathbb{P}$-generic over M. Show that there is an $\in$-formula φ and $\mathbb{P}$-names $\langle \dot{x}_i \mid i < k \rangle$ for some $k \in \omega$, such that $M[F] \models \varphi(\dot{x}_0^F, \ldots, \dot{x}_{k-1}^F)$, however there is no $p \in F$ such that $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{k-1})$.

Problem 27 [5 points] Let $\mathbb{P} = \langle P, \leq, \ldots \rangle$ be a partial order. Given $A \subseteq P$, we say that

$$a = \sup A$$

in case $a \in P$ is the least upper bound of A in $\mathbb{P}$, that is

- $b \leq a$ for every $b \in A$ and
- $a \leq c$ whenever $c \in P$ is such that $b \leq c$ for every $b \in A$.

For a Boolean algebra $\mathbb{B}$ and some $A \subseteq B$, $\sup A$ may or may not exist in B. We say that $\mathbb{B}$ is *complete* in case $\sup A$ exists in B for every $A \subseteq B$. If $\mathbb{P}$ is any separative partial order, we say that $\mathbb{B}$ is a *completion* of $\mathbb{P}$ in case P is a dense subset of $B \setminus \{0\}$ and $\mathbb{B}$ is a complete Boolean algebra.

- Provide a definition of when $a = \inf A$, and show that if $\mathbb{B}$ is a Boolean algebra, then $\inf A$ exists in B for every $A \subseteq B$ if and only if $\sup A$ exists in B for every $A \subseteq B$.
- Show that if $\mathbb{B}$ and $\mathbb{C}$ are both completions of a given separative partial order $\mathbb{P}$, then $\mathbb{B}$ and $\mathbb{C}$ are isomorphic.

Problem 28 [8 points] Let $\mathbb{P}$ be a separative partial order. We say that $A \subseteq P$ is a *cut* in case $q \in A$ whenever $q \leq p$ for some $p \in A$. For $p \in P$, let $A_p = \{x \mid x \leq p\}$. A cut A is *regular* in case

$$p \notin A \text{ implies } \exists q \leq p \; A_q \cap A = \emptyset.$$

- Show that A_p is a regular cut whenever $p \in P$.
- Show that each cut A is included (as a subset) in a subset-least regular cut $\bar{A}$.
- Let B be the collection of regular cuts of P. Define the operations of $\mathbb{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ as follows.
  - $0 = \emptyset$,
  - $1 = P$,
  - $b \wedge c = b \cap c$,
  - $b \vee c = \overline{b \cup c}$, where $\overline{b \cup c}$ denotes the subset-least regular cut containing $b \cup c$, and
  - $\neg b = \{c \mid A_c \cap b = \emptyset\}$.

  Show that $\mathbb{B}$ is a complete Boolean algebra.
- Show that P can be identified with a dense subset of $B \setminus \{0\}$ via the embedding that maps $p \in P$ to $A_p \in B$.

Note: We have shown that if $\mathbb{P}$ is any separative partial order, then there is a unique complete Boolean algebra $\mathbb{B}$ such that P is a dense subset of $B \setminus \{0\}$.